OUTLINE

- An illustrative example
- Mathematical behavioral science; Theories
- Game theory: some history
- Game theory explained
- Evolutionary game theory
The most famous game theory example is the **Prisoner’s Dilemma** (Dresher & Flood, Tucker; Raiffa c. 1950) and it is an excellent illustrative example.

Suppose we have two thieves, Liam and Mai. These sibling partners in crime have been captured by police and placed in separate cells. Liam and Mai are offered the opportunity to confess to the crime.

- If both Liam and Mai confess, then they will both spend five years in prison.
- If neither confesses, then they will both spend 1 year in prison.
- If only one confesses, then the confessor goes free and the nonconfessor spends ten years in prison.

If you are Mai, what do you do?
Are the behavioral and social sciences science?

- Anthropology, Biology, Cognitive Science, Economics, Linguistics, Political Science, Psychology, Sociology

- By economics, consider only microeconomics, which is essentially the theory of prices, i.e. how supply and demand interact in various markets.

To be a science (minimally), the scientific theory must be

- explanatory
- predictive
- mathematical
When mathematics was first used in the social sciences, there was the tendency to take the methods and models that worked in physics and attempt to apply them directly to social and behavioral problems. This did not work well, it lead to some bad math and bad science.

However, beginning in the 1940’s (more or less), social scientists began to create mathematical methods and models specifically for the social and behavioral sciences.

Mathematical behavioral science might seem to be an oxymoron unlike say mathematical physics.

However, recall that it was not until the 1600’s that there was classical mechanics. And not until the early 1900’s was there the discovery of relativity and quantum mechanics.

The physical world’s structure, as opposed to the social and behavioral world’s structure, is much easier to determine through observation and experimentation.

Thus, it should not surprise us that the mathematical behavioral sciences have made only modest gains in the past 60 years or so.
Primary Fields of the mathematical behavioral sciences

- Measurement theory (Social & Behavioral Science)
- Mathematical Psychology, Psychometrics, Cognitive Models (Cognitive Science, Psychology)
- Social Network Theory (Anthropology, Sociology)
- Game Theory, Decision Theory (Biology, Cognitive Science, Economics, Philosophy)
ECONOMICS AS SCIENCE

From the time of Adam Smith’s *The Wealth of Nations* (1776) until the present, economics has fared quite well with explaining how economies work.

The mathematization of economics did not begin to occur until the late 1800’s and was not in place as the way to do economics until after World War II (neo-classical economics).

So economics can explain and it is mathematical.

The latter we will discuss in more detail shortly.

But can economics predict?

While economics can not predict in the same way that physics can, it has more predictive power than it if often given credit for.

This will be clear through some of the economic examples that game theory can be applied to.
While the focus on this presentation is only on one aspect of mathematization of economics, viz. game theory. There are other subfields of economics that since the late 1800's has become very mathematical.
Two books


Differential calculus

- For example: the *marginal revenue function* is the first derivative of the *revenue function*; likewise for the *profit* and *marginal profit function*. 
MATHEMATICS & ECONOMICS

- Linear algebra (matrix theory)
  - E.g., to represent an input and output model
- Econometrics
  - Is the unification of statistics, economic theory, and mathematics (Ragnar Frish, 1895-1973, co-winner of the first Nobel Prize in economics in 1969)
- Bayesian (or subjectivist) statistics and probability theory
Two ways that we will use the word *theory*.

- In mathematics, theory means a branch of mathematics that has a set of axioms and a common subject matter, e.g. set theory, group theory, game theory.

- Thus, these theories are correct by virtue of their being mathematics since theorems are proved.

- The content of the theorems is about mathematical patterns and structures.
In science, a theory is a coherent *mathematical* model of natural phenomena, i.e. empirical facts, that both explains and predicts. For example, the theory of relativity, string theory, atomic theory, **the theory of evolution**.

Until the seminal work by John Maynard Smith, the non-mathematical nature of the theory of evolution was worrisome.
Why is mathematics able to model the empirical world?

If the math is assumed to be empirical, how do we know it is true?

If the math is *a priori*, how does it model the physical world?

A brief, truncated history of formal game theory (1700s-1927)

- 1731 James Waldegrave provided an analysis and a solution to a certain type of two person game.

- 1838 Augustin Cournot’s *Researches into the Mathematical Principles of the Theory of Wealth* discusses utility and an equilibrium in noticeable game theoretic fashion.

- 1913 The logician Ernst Zermelo’s stated a theorem about chess (A player is able to force: Black wins, White wins, or draw).

- 1921-7 The analyst Emile Borel gave the first modern formulation of strategic games in a series of papers.
A brief, truncated history of formal game theory (1928-1950s)

- **1928** John von Neumann proves the minmax theorem and introduces the extensive form of the game.

- **1944** von Neumann and Oskar Morgenstern publish *Theory of Games and Economic Behavior*.

- **1950-53** John Nash (of *A Beautiful Mind* fame) publishes four papers about strategies and equilibria in bargaining and noncooperative games.

- **1957** R. Duncan Luce & Howard Raiffa, *Games and Decisions*
WHAT IS GAME THEORY

- Game theory is the mathematical analysis of strategic interactions between individuals that produce outcomes.

- These outcomes are judged by their utility, i.e. their worth, to the individuals.

- Originally game theory came from the study of pallor games and economics by mathematicians, e.g. Zermelo, Borel, von Neumann, Nash

- There are now a plethora of applications of game theory to a number of fields.

Thus, game theory is one of the most widely used mathematical paradigms in the sciences.

Today we focus on three applications of evolutionary game theory:

- Sex Ratio
- Justice
- Language

But before we turn to evolutionary game theory, let’s talk about standard game theory and the Prisoner’s Dilemma.
INFORMAL DEFINITIONS

- Two important (informal) definitions.

- A strategy describes the behavior of an individual given a particular situation, i.e. a “game.”

- A (Nash) equilibrium is when in a game of two or more people, if they play a strategy that happens to be a Nash Equilibrium, then neither person can deviate from this strategy and increase his payoff.
PRISONER’S DILEMMA

❖ The thieves Liam and Mai are offered the opportunity to confess to the crime.

❖ If both Liam and Mai confess, then they will both spend five years in prison.

❖ If neither confesses, then they will both spend 1 year in prison.

❖ If only one confesses, then the confessor goes free and the nonconfessor spends ten years in prison.
So given the scenario, what is the dilemma?

- It is better to not confess, if you both don’t confess.
- It is better to confess, if the other does not.
- But without a way to communicate with your partner in crime, what is your strategy?

<table>
<thead>
<tr>
<th></th>
<th>Liam</th>
<th>Mai</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No confession</td>
<td>Confession</td>
</tr>
<tr>
<td>No confession</td>
<td>(1,1)</td>
<td>(10,0)</td>
</tr>
<tr>
<td>Confession</td>
<td>(0,10)</td>
<td>(5,5)</td>
</tr>
</tbody>
</table>

The table is called the strategic form of the game.
Of course, the outcome that we desire is that we both don’t confess.

But what would you do?

Mai and Liam have the same thought: no matter what my partner does it is better for me to confess.

For Mai, her expected value (i.e., the mean) for the first row is 5.5 and the second row is 2.5. Likewise for the two columns for Liam.

<table>
<thead>
<tr>
<th></th>
<th>Liam</th>
<th></th>
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<tbody>
<tr>
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</table>

Thus the equilibrium is (5,5).
Technical notes.

Cooperating (not confessing) is *strictly dominated* by defecting (confessing).

The equilibrium at (5,5) is a *Nash Equilibrium*.

However, (5,5) is not *Pareto optimal* since (1,1) is the optimal solution.

The tree is called the *extended form* of the game.
PRISONER’S DILEMMA

- Technical notes.

- Backwards induction (von Neumann)

- Start at the bottom of the tree. If you are Liam and you pick “don’t confess,” you know Mai would pick “confess.” Likewise, if you pick “confess,” then Mai will pick “confess.”

So you know that since Mai will always play “confess,” Liam plays “confess” as well.
WEAKNESSES OF THE PROPOSED THEORY

- While Game Theory in this form, “classical game theory,” has proved to be very useful, there are some weaknesses to the theory proposed so far.

- Assumes players are “rational” and only looks at individuals, not populations.

- The classical theory is a static theory.
STATIC VS. DYNAMIC

- The example of the Prisoner's Dilemma and the discussion of game theory has thus far been static. In other words, the discussion has not looked at the underlying process by which behaviors or strategies change.

- von Neumann and Morgenstern noted that their theory was static, but felt a non-static, i.e. dynamic approach, would be fruitful, but only after the static approach had been thoroughly investigated.

- So our desire would be to have a game theory that is dynamic and also does not assume "rationality" and is able to look at changes in populations, whether biological, socio-economic, or cultural.
Happily there is a dynamic scientific theory that allows for such dynamic ideas as replication, recombination, mutation, and migration and as such looks at populations and does not presuppose an *a priori* notion of rationality.

This theory is the *theory of evolution*.

We will see how this works by looking at a simple biological example.

This kind of game theory is called *evolutionary game theory*.
A brief, truncated history of evolutionary games

- **1930** R.A. Fisher’s *The Genetic Theory of Natural Selection* (statistician, biologist)

- **1961** R.C. Lewontin “Evolution and the Theory of Games” (biologist)

- **1969** David Lewis, *Convention* (philosopher)

- **1972** John Maynard Smith “Game Theory & the Evolution of Fighting”

- **1973** Maynard Smith and George Price “The Logic of Animal Conflict” (biologists)
A brief, truncated history of evolutionary game theory, continued

- 1982 Maynard Smith *Evolution and the Theory of Games*

- 1980-1983 Reinhard Selten’s papers on evolutionary game theory and economics (economist; received the Noble Prize in 1994 with J. Nash and J. Harsanyi)

- 1984 Robert Axelrod *The Evolution of Cooperation* (political scientist)
FISHING FOR SEX (RATIO)

- Fisher wanted to answer the following biological question definitively, which scientists have not been able to answer, to include Darwin:
  - Why is the sex ratio of mammals in general, to include harem-forming species, close to 1/2?
  - That is, why is there roughly 50% males, 50% females even with, say gorillas, where only a small portion of the males actually mate?

- While Fisher did not explicitly use evolutionary game theory to answer his question, it can be viewed through the evolutionary game theory paradigm.
SEX RATIO

Every child has a male parent and a female parent, where it gets half of its genes from each.

Suppose there were a preponderance of females in the population.

Then males would have more children on average than females and would contribute more genes to the next generation.

An individual who carried a tendency to produce more males would have a higher expected number of grandchildren than the population average, and that genetically based tendency would spread through the population.

Likewise, in a population with a preponderance of males, a genetic tendency to produce more females would spread.
Thus, there is an evolutionary feedback that tends to stabilize at equal proportions of males and females.

This argument works even if there is a large proportion of males that never breed, e.g. 90% of male cattle are eaten before having a chance to breed, evolutionary pressures will still drive the sex ratio to unity.

This evolutionary stable proportion is called a evolutionary stable equilibrium since it is the optimal reproduction “strategy.”
There is a mathematical model of differential replication called the replicator dynamics.

This is a nonlinear differential equation that captures the evolutionary notion of selection, where there is more than one type in the population.

Thus the mathematics of evolutionary game theory falls under nonlinear dynamics, or as it is popularly known chaos theory.

\[ x_i(t) \] denotes the frequency of type \( i \)
\[ f_i \] denotes the fitness of type \( i \)
\[ \phi \] denotes the average fitness of the population

\[ \phi = \sum_{i=1}^{n} x_i f_i \]
\[ \dot{x}_i = x_i (f_i - \phi), \quad i \in \mathbb{N} \]

Thus, the frequency of type \( i \) increases, if its fitness exceeds the average fitness of the population. Otherwise it will decline.
Unsurprisingly the replicator dynamics is related to the Lotka-Volterra equations which are used to model populations, e.g. to model predator-pray dynamics.

A. Lotka based his work on autocatalytic chemical reactions (1910) on the work of A. Kolmogorov and V. Volterra studied commercial fishing in the Adriatic (1926).

There is an extension of the replicator dynamics (and the quasispecies equation of evolution), which takes into account mutation, this is called the replicator-mutator dynamics.

\[ \dot{x}_i = \sum_{j=1}^{n} x_j f_j q_{ji} - \phi x_i \]

\( q_{ij} \) is the matrix that represents that that type \( i \) mutates to type \( j \).
It turns out that the same analysis works for non-biological examples as well such as those in the social and behavioral sciences.

Let us look at what evolutionary game theory can tell us about the evolution of justice.

And we’ll see that this example is analogous to the male-female sex ratio example from biology.
The Fair Division Game, or how to cut a cake

Two players- Mai and Liam- are to divide a red velvet cake.

Their positions are symmetric, e.g. if Mai gets 40%, then Liam will have 60%.
RULES OF THE GAME

- Each player writes the percentage of the cake that he desires on a sheet of paper, folds it, and hands it to a referee (me).

- If the total is more than 100%, the referee gets the cake.

- Otherwise, Liam and Mai get what they claim.

- If they claim less than a 100%, then we suppose that the referee gets the difference.
We expect that most people would write 50%, so 50-50 would be a strict Nash equilibrium.

But it is not the only strict Nash equilibrium since there are actually an infinite number of Nash equilibria, e.g. 1/3-2/3, 70-30, and so on.

While 40-40 would be an equilibrium, it would not be a strict equilibrium since both Liam and Mai could do better.

This is a static game like the Prisoner’s Dilemma.
In order to look at this game as a way to think about justice from the evolutionary perspective, then we need to treat this game dynamically.

Individuals, paired at random from a large population, play the *bargaining game*.

The cake represents a quantity of Darwinian fitness (expected number of offspring) that can be divided and transferred.

Individuals reproduce, on average, according to their fitness and pass along their strategies to their offspring.
Suppose we have a population of individuals demanding 60% of the cake. Meeting each other they get nothing.

If anyone were to demand a positive amount less than 40% he would get that amount and thus do better than the population average.

Likewise, for any population of individuals that demand more than 50% (and less than 100%).

Suppose we have a population demanding 30%. Anyone demanding a bit more does better than the population average.

Likewise for any amount less than 50% (and greater than 0).
This means that the only (pure) strategies that are equilibria Demand 50\% and Demand 100\%.

However, Demand 100\% is unstable. In a population in which everyone demands 100\%, everybody gets nothing, and if a mutant popped up who made a different demand, he would still get nothing.

Yet, if a modest proportion of mutants arose who demanded, say, 45\%, then most of them would be paired with 100 percenters and get nothing, but when they met one another they would get 45\%.

On average their payoff would be higher than that of the population, increasing their numbers, so Demand 100\% is easily invaded by mutants.
DYNAMIC CAKE CUTTING

- Demand 50% is a stable equilibrium.
- In a population in which everyone demands half the cake, any mutant who demanded anything different would get less than the population average.
- So demanding 50% of the cake is an evolutionary stable strategy.

This triangle is called a simplex and it is one face of a three dimensional figure (an equilateral pyramid).

Thursday, September 16, 2010
In terms of nonlinear dynamics (replicator-mutation equation), this is an attracting dynamical equilibrium.

There are two basins of attraction at 50-50.

Brian Skyrms’ simulator.

Previous simplex was 50 runs, this is 200.
EVOLUTION OF LANGUAGE

- Language is an unlimited *replication* device for cultural information.

- Since it is a replication device, it lends its self to evolutionary analysis.

- Further, since you can view individuals as playing a game when a language is being created, then one can try to determine how language evolves by using evolutionary game theory.
Once again think back to the sex ratio and cake cutting examples as well as historical events such as the Norse invasion on England.

The mutant invaders, e.g. the Norse, brought their own language with them and as such changed the native language, and allowed for the creation of Anglo-Saxon.

This can be modeled by the replicator-mutator dynamics.

The dynamics explains how words get meaning and how words change even from generation to generation.

Related to game theory is the study of decisions, which is called decision theory.

- Duncan Luce, *Individual Choice Behavior*, 1959
REFERENCES

- Some additional (non-historical) game theory references
  - Binmore, K., *Fun & Games*, 1992
REFERENCES

- Some additional (non-historical) game theory references
Questions, comments,...

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Gratias.