Calculus I: Tests for Local Extrema and Concavity

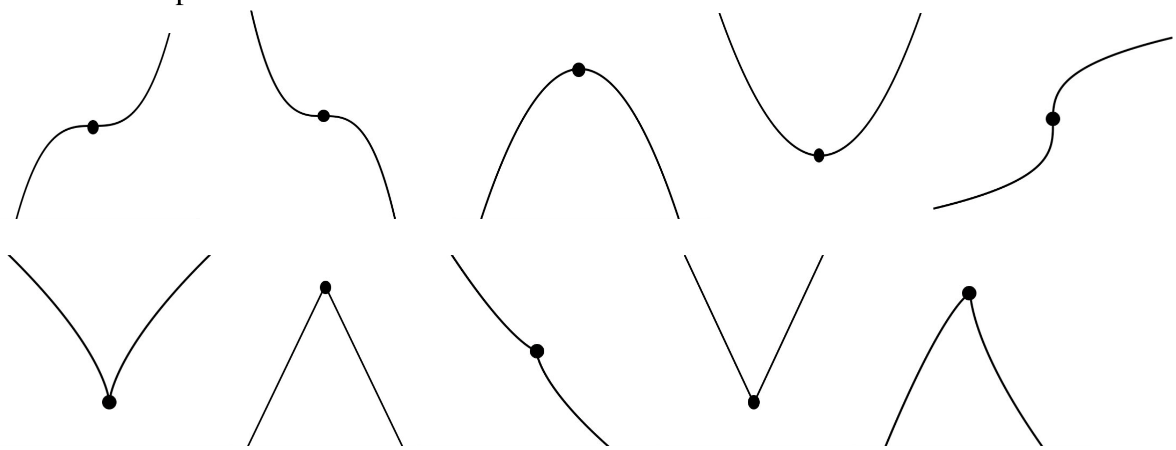
In all of these problems, each function *f* is **continuous** on its domain. This means the graph of *f* has no jumps, breaks, or holes in it. In other words, you can draw the graph of *f* without lifting your pen or pencil.

1) For a function *f*, a **critical "point"** or **critical number** is a number $x = c$ for which $f'(c) = 0$ or $f'(c)$ is undefined or does not exist (DNE for short). Find the critical points of each of the following functions.

	Critical point(s):	Reason:
(a) $f(x) = x^3 - 6x^2 + 9x + 8$	$x = 1$	$f'(1) = 0$
	$x =$	
(b) $f(x) = x^4 - 4x^3 + 20$		
(c) $f(x) = x^4$		
(d) $f(x) = \sqrt[3]{x}$		

2) If $x = c$ is a critical point for the function *f* with $f'(c) = 0$, then the graph of *f* has slope 0 at the point $(c, f(c))$. If $x = c$ is a critical point for the function *f* with $f'(c)$ undefined, then the graph of *f* has a sharp corner, a cusp, or a vertical tangent at the point $(c, f(c))$.

Possible shapes for the graph of *f* near the point $(c, f(c))$ include the following graphs. In each graph, the point marked with a dot is the point $(c, f(c))$. Label this point "min" for local (or relative) minimum point, "max" for local (or relative) maximum point, or "neither" if the point is neither a local minimum point nor a local maximum point.



(a) For each graph above, determine if $f'(x)$ is positive or negative for $x < c$. Determine if $f'(x)$ is positive or negative for $x > c$. You may mark + or – on each side of the point $(c, f(c))$, if you like.

(b) Use the results of part (a) to write a three-part rule for using the derivative f' to determine if a given critical point $x = c$ is the *x*-coordinate of a local (or relative) minimum point, a local (or relative) maximum point, or neither – that is, if $(c, f(c))$ is a local minimum point, a local maximum point, or neither.

If $f'(x)$ changes from ____________ to ____________ at $x = c$,
then the point $(c, f(c))$ is a local minimum point on the graph of *f*.
We also say the function *f* has the local minimum value $f(c)$ at $x = c$.

If $f'(x)$ changes from ____________ to ____________ at $x = c$,
then the point $(c, f(c))$ is a local maximum point on the graph of *f*.
We also say the function *f* has the local maximum value $f(c)$ at $x = c$.

If $f'(x)$ does not ______________ ______________ at $x = c$,
then the point $(c, f(c))$ is neither a local minimum point nor a local maximum point on the graph of *f*.

This rule is called the **First Derivative Test for Local Extrema**.

Again, the rule you wrote on the preceding page is called the **First Derivative Test for Local Extrema**. (Local extrema are local minimum and maximum values, also called local minima and maxima.)

(c) Use the **First Derivative Test for Local Extrema** from part (b) to determine if each critical point $x = c$ from Problem 1 corresponds to a local minimum point, a local maximum point, or neither. Use the derivative $f'(x)$ of each function *f* and a sign chart for $f'(x)$, rather than the graph of the function *f*. (You may use the graph of *f* to check your answers.)

$f(x) = x^3 - 6x^2 + 9x + 8$

Conclusion:
From Exercise 1(a), the critical points for f are $x = 1$ and $x = 3$. The point $(1,12)$ is a local maximum point on the graph of f because f' changes from positive to negative at $x = 1$. The point $(3,8)$ is ….

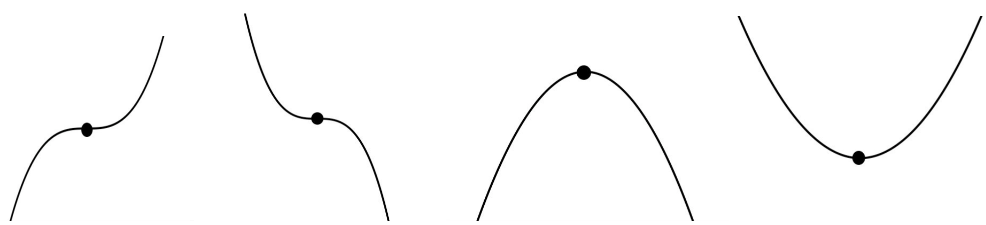
$f(x) = x^4 - 4x^3 + 20$

$f(x) = x^4$

$f(x) = \sqrt[3]{x}$

3) Very few of the graphs we encounter in calculus class have vertical tangent lines and even fewer have sharp corners or cusps. This means most of the critical points we find will be critical points $x = c$ at which $f'(c) = 0$. If $x = c$ is a critical point for the function *f* with $f'(c) = 0$, then the graph of *f* has slope 0 at the point $(c, f(c))$. Possible shapes for the graph of *f* near the point $(c, f(c))$ include the following graphs. In each graph, the point marked with a dot is the point $(c, f(c))$.

(a) For each graph below, determine if the graph is concave up, concave down, or neither at the point $(c, f(c))$. Is $f''(c)$ positive, negative, or zero at $x = c$?



Concavity at $x = c$				
Sign of $f''(c)$				

(b) Use the results of part (a) to write a three-part rule for using the second derivative f'' to determine if a given critical point $x = c$ is the *x*-coordinate of a local (or relative) minimum point or a local (or relative) maximum point.

If $f'(c) = 0$ and $f''(c)$ _______________, then $(c, f(c))$ is a local minimum point on the graph of *f*.

If $f'(c) = 0$ and $f''(c)$ _______________, then $(c, f(c))$ is a local maximum point on the graph of *f*.

If $f'(c) = 0$ and $f''(c)$ _____________ or $f''(c)$ is not defined, then we can draw no conclusion from this test.* In order to determine if $(c, f(c))$ is a local minimum point, a local maximum point, or neither on the graph of *f,* we must apply another test, such as the First Derivative Test.

This rule is called the **Second Derivative Test for Local Extrema** (local minimum and maximum values). Note that it is <u>not</u> a test for concavity, but rather uses what you already know about the relationship between concavity and the second derivative to determine local minimum and maximum values.

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* Don't believe it? Compare $f(x) = x^3$ and $f(x) = x^4$ at $x = 0$ ($c = 0$), or do part (c).

Again, the rule on the preceding page is called the **Second Derivative Test for Local Extrema**.

(c) Use the **Second Derivative Test for Local Extrema** from part (b) to determine if each critical point $x = c$ from Problem 1, parts (a), (b), and (c) only, corresponds to a local minimum point or a local maximum point, or if no conclusion can be drawn from the Second Derivative Test. Use the second derivative $f''(x)$ of each function, rather than the graph of the function f. (You may use the graph of f to check your answers.)

In the two cases in which no conclusion can be drawn, what result did the First Derivative Test give? (See Problem 2, part (c).)

$f(x) = x^3 - 6x^2 + 9x + 8$

Conclusion:
From Exercise 1(a), the critical points for f are $x = 1$ and $x = 3$. The point $(1,12)$ is a local maximum point on the graph of f because $f'(1) = 0$ and $f''(1) < 0$. The point $(3,8)$ is ….

$f(x) = x^4 - 4x^3 + 20$

$f(x) = x^4$

4) (a) For each of the functions below, find all numbers $x = c$ for which $f''(c) = 0$ or $f''(c)$ is undefined.

$f(x) = x^4 - 4x^3 + 20$

$f(x) = x^4$

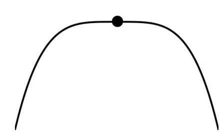
$f(x) = \sqrt[3]{x}$

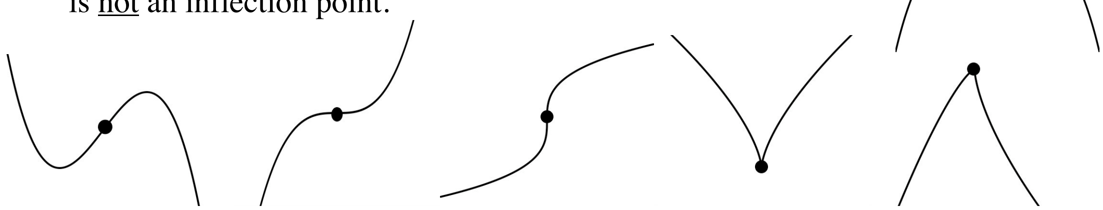
$f(x) = \sqrt[3]{x^2}$

(b) An *inflection point* is a point on the graph of a function at which the graph changes concavity. Four of the points shown on the graphs below are inflection points; two are not. Write a two-part **Second Derivative Test for Inflection Points**: For an x-coordinate $x = c$ for which $f''(c) = 0$ or $f''(c)$ is undefined (examples of graphs containing points with such x-coordinates are shown below),

if $f''(x)$ __________ __________ at $x = c$, then $(c, f(c))$ is an inflection point;

if $f''(x)$ does not _________ ________ at $x = c$, then $(c, f(c))$ is not an inflection point.





(c) For each number $x = c$ from part (a), use the **Second Derivative Test for Inflection Points** to determine if the point $(c, f(c))$ is an inflection point or not. Use the second derivative $f''(x)$ of each function f and a sign chart for $f''(x)$, rather than the graph of the function f. (You may use the graph of f to check your answers.)

5) Only two of the following six statements are true. For each of the four false statements, give an example of a function $f(x)$ and an x-value $x = c$ *from this worksheet* that violates the statement. For instance, you might write: "False: The function $f(x) = \sqrt[3]{x}$ from Problem 4, part (c), violates the statement at $x = 0$."

(a) True or false? If the point $(c, f(c))$ is a local minimum point or maximum point on the graph of $y = f(x)$, then it always is true that $x = c$ is a critical point of *f*.

(b) True or false? If $f'(c) = 0$, then the point $(c, f(c))$ is a local minimum point or maximum point on the graph of $y = f(x)$

(c) True or false? If $f'(c)$ is undefined, then the point $(c, f(c))$ is a local minimum point or maximum point on the graph of $y = f(x)$.

(d) True or false? If the point $(c, f(c))$ is an inflection point on the graph of $y = f(x)$, then it always is true that $f''(c) = 0$ or $f''(c)$ is undefined.

(e) True of false? If $f''(c) = 0$, then the point $(c, f(c))$ is an inflection point on the graph of $y = f(x)$.

(f) True or false? If $f''(c)$ is undefined, then the point $(c, f(c))$ is an inflection point on the graph of $y = f(x)$.

6) **Preview!** For the function $f(x) = x^4 - 4x^3 + 20$ from Problem 1, part (b), the point $(3,-7)$ is the global (or absolute) minimum point on the graph of *f* and the value $f(3) = -7$ is the global (or absolute) minimum value for the function *f*. Use the derivative $f'(x)$ of $f(x) = x^4 - 4x^3 + 20$ to convince me or your neighbor (in writing, of course) that the point $(3,-7)$ is the global minimum point on the graph of *f*. How is your argument in this problem different from using the First Derivative Test for Local Extrema to show that the point $(3,-7)$ is a local minimum point on the graph of *f*?

2) (d) Why does the First Derivative Test for Local Extrema work?
For a continuous function f and a critical point $x = c$ for f,
if the derivative $f'(x)$ changes from negative to positive at $x = c$, then the

slope of the graph of f changes from ____________ to _____________ at $x = c$.

Hence, the graph of f changes from ________________ to ________________

at $x = c$, so that $(c, f(c))$ is a local ________________ point on the graph of f.

If the derivative $f'(x)$ changes from positive to negative at $x = c$, then the

slope of the graph of f changes from ____________ to _____________ at $x = c$.

Hence, the graph of f changes from ________________ to ________________

at $x = c$, so that $(c, f(c))$ is a local ________________ point on the graph of f.

If the derivative $f'(x)$ does not change sign at $x = c$, then the

slope of the graph of f _________________________ at $x = c$.

Hence, the graph of f does not change ___________________ at $x = c$,
so that $(c, f(c))$ is neither a local minimum point nor a local maximum point
on the graph of f.

3) (d) Why does the Second Derivative Test for Local Extrema work?
For a twice differentiable function f and a critical point $x = c$ for f for which
$f'(c) = 0$ (for a twice differentiable function, there are no critical points $x = c$
for which $f'(c)$ does not exist), the graph of f at (and near) the point $(c, f(c))$
has one of the four shapes shown in Problem 3(a) or it is a horizontal line.

If $f''(c) > 0$, then the graph of f is _____________________ at (and near) the
point $(c, f(c))$. This eliminates four of the five possible shapes, leaving one in

which the point $(c, f(c))$ is a local ________________ point on the graph of f.

If $f''(c) < 0$, then the graph of f is _____________________ at (and near) the
point $(c, f(c))$. This eliminates four of the five possible shapes, leaving one in

which the point $(c, f(c))$ is a local ________________ point on the graph of f.

4) (d) Why does the Second Derivative Test for Inflection Points work?
For a continuous function f and a value $x = c$ for which $f''(c) = 0$ or $f''(c)$ is
undefined, if the second derivative $f''(x)$ changes sign at $x = c$, then the graph

of f changes __________________ at $x = c$, so that the point $(c, f(c))$ is an
inflection point on the graph of f. If the second derivative $f''(x)$ does not

change sign at $x = c$, then the graph of f does not change __________________
at $x = c$, so that the point $(c, f(c))$ is not an inflection point on the graph of f.